The birational geometry of GIT quotients

(w/ R. Reboulet)

Geometric Invariant Theory (GIT), Mumford '60s, is the quotient theory in algebraic geometry. Interesting for moduli theory.

- X proj variety $/\mathbb{C}$
- L ample line bundle
- G reductive group (complexification of a maximal compact subgroup) e.g. $U(n) \subset GL(n)$

Assume $G \curvearrowright (X, L)$
(so the action lifts to a linear action on L)

<u>Goal</u> Construct a "reasonable" quotient of (X, L) by G.

Two equivalent approaches: cover X by affines, or work globally.

Affine approach: (i) locally consider $U \subset X$ affine

$U = X - V(s)$, $s \in H^0(X, L)^G$, consider

$U /\!/ G = \operatorname{Spec}(A^G)$, where $\operatorname{Spec} A = U$.

so regular functions on $U/\!/G$ are just G-invariant functions. Important: A^G finitely generated.

(ii) Glue the $U/\!/G$ across different charts to produce $X /\!/_L G$.

(iii) Projective approach: $X/\!\!/_L G = \mathrm{Proj}\left(\bigoplus_{k\geq 0} H^0(X, kL)^G\right)$.

<u>Basic properties</u>: • $X/\!\!/_L G$ projective (proper, separated)

• Not immediately clear what ($\mathbb{C}$-) points of $X/\!\!/_L G$ are.

Def We say $x \in X$ (or $G.x$) is

(i) semistable if $\exists$ $s \in H^0(X, kL)$ with $s(x) \neq 0$.

(ii) polystable if x is semistable & $G.x$ is closed in X^{ss}

(iii) stable if x is polystable & G_x finite.

$X^{ss} \subset X$ is Zariski open. We have a rational map $X \dashrightarrow X/\!\!/_L G$, defined on X^{ss}.

$$X^{ss} \to X/\!\!/_L G.$$

The ($\mathbb{C}$-)points of $X/\!\!/_L G$ are precisely in bijection with polystable orbits ($/\mathbb{C}$).

Unstable points are ignored!

Variation of GIT

Q What is the dependence of $X/\!\!/_L G$ on L?

Thm (Dolgachev–Hu, Thaddeus) '90s Consider the ample cone $\mathrm{Amp}_{\mathbb{Q}}(X)^G$ such that $\exists$ stable point. Then:

(i) any two GIT quotients of X are birational.

(ii) only finitely many birationally equivalent GIT quotients appear.

(iii) the birational transformations are (Thaddeus) flips.

Interlude on moduli theory

GIT is built to construct moduli spaces (projective varieties, sheaves, maps). Roughly:

(i) "embed all objects in some large variety/scheme" (e.g. embed all varieties in some projective space)

(ii) "check orbits for some group action are isomorphism classes of objects" (e.g. $GL(n+1) \curvearrowright \mathbb{P}^n$)

(iii) "take GIT quotient of some parameter space of objects" (e.g. $\mathrm{Hilb}_{\mathbb{P}^n} /\!/_{L_{Hilb}} G$)

This process works for $\overline{M}_g$, $\overline{M}_{g,n}$, moduli of stable vector bundles over a smooth proj curve, Kontsevich's stable maps...

In higher dim, GIT is usually a motivational philosophy, not so much a tool (e.g. K-stability, Bridgeland stability conditions on triangulated categories).

The analogue of VGIT in moduli theory is wall-crossing of moduli spaces: the line bundle is a "stability condition" & varying the stability condition varies the moduli space birationally.

<u>E.g.</u> Varying coefficients in $\overline{M}_{g,n}$, Bridgeland stability conditions.

But GIT produces finitely many birational models: how to obtain more?

Universal variation of GIT

We ask how the GIT quotient depends on X, as a counterpart to VGIT.

Def A GIT quotient over (X, G) is a GIT quotient $Y /\!/_{L_Y} G$, where $\pi : Y \to X$ is a G-equivariant birational morphism. $G \curvearrowright (Y, L_Y)$

Thm (B.-Reboulet) GIT quotients over (X, G) are all mutually birational, and form a projective system. The projective limit is isomorphic to the Riemann-Zariski space of $X /\!/_L G$ for any L.

<u>Here</u> : • To say that these form a projective system means that for any

$$\begin{tikzcd} & Z/\!/_{L_Z} G \arrow[dl] \arrow[dr] & \\ Y_1/\!/_{L_1} G \arrow[rr, dashed] & & Y_2/\!/_{L_2} G \end{tikzcd}$$

we find a $Z/\!/_{L_Z} G$ with a morphism to both.

- The Riemann-Zariski space of a proj var W is the projective limit of <u>all</u> birational models of W. It is a locally ringed space.
- Two proj varieties are birational
  $\iff$ their RZ spaces are isomorphic.

<u>Cor</u> GIT quotients over (X, G) capture "all" of the birational geometry of $X/\!/_{L} G$.

Moduli theory: brief return

Our work suggests it's interesting to vary both the "stability condition" = ample line bdl & the "parameter space" = X itself (birationally)

<u>Q</u> What does this mean? Does this work for Bridgeland stability conditions? What are birational transformations of triangulated categories? E.g. $D^b \mathrm{Coh}(X)$?

Ideas in the proof

The proofs are fairly straight forward.

<u>Prop</u>$^{(D.R)}$ Let $Y_1 /\!/_{L_1} G$, $Y_2 /\!/_{L_2} G$ be GIT quotients over (X, G). Then they are birational.

<u>Pf</u> $\exists$ x_1, x_2 stable in Y_1, Y_2, so let $x_1 \in U_1 = Y_1 - V(s_1)$, $x_2 \in U_2 = Y_2 - V(s_2)$

Then $U_1 /\!/ G$ is Zariski dense in $Y_1 /\!/_{L_1} G$, $U_2 /\!/ G$ similarly

As Y_1, Y_2 are birational to X, $\exists$ $U \subset X$ embedding in U_1, U_2 & then $U /\!/ G \subset U_1 /\!/ G$ as a Zariski dense subset.

$\boxtimes$

Thm One obtains a proj system.

Pf Need to find

$$\begin{tikzcd} & Z/\!/_{L_2} G \arrow[dl] \arrow[dr] & \\ Y_1/\!/_{L_1} G \arrow[rr, dashed] & & Y_2/\!/_{L_2} G \end{tikzcd}$$

By birationality set $W \subset Y_1/\!/_{L_1} G$ s.t.

$$\begin{tikzcd} Bl_W(Y_1/\!/_{L_1} G) \arrow[r] \arrow[d] & Y_2/\!/_{L_2} G \\ Y_1/\!/_{L_1} G. & \end{tikzcd}$$

By Kirwan, Reichstein $Bl_W(Y_1/\!/_{L_1} G) \cong (Bl_{\tilde{W}} Y_1)/\!/_{L} G$ $\boxtimes$